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Linearized Gravity in the Karch-Randall Braneworld

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Abstract

We present a linearized gravity investigation of the bent braneworld, where an AdS_4 brane is embedded in AdS_5 . While we focus on static spherically symmetric mass distributions on the brane, much of the analysis continues to hold for more general configurations. In addition to the identification of the massive Karch-Randall graviton and a tower of Kaluza-Klein gravitons, we find a radion mode that couples to the trace of the energy-momentum tensor on the brane. The Karch-Randall radion arises as a property of the embedding of the brane in the bulk space, even in the context of a single brane model.

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1. Introduction.

In the Randall-Sundrum model [1], gravity is trapped on a (flat) 3-brane embedded in an uncompact five-dimensional spacetime. In particular, a linearized treatment of gravity indicates the presence of a massless graviton localized on the brane, along with a tower of Kaluza-Klein modes in the bulk [1–4]. From a four-dimensional perspective, the massless graviton yields an ordinary $1/r$ Newtonian potential, while the Kaluza-Klein modes produce a $1/r^3$ correction, so that [1,2,5]

$$V(r) = \frac{Gm_1m_2}{r} \left(1 + \frac{2L_5^2}{3r^2} \right), \quad (1.1)$$

where L_5 is the AdS_5 “radius”.

In a complementary approach, corrections to the Newtonian potential may also be obtained through the investigation of braneworld black holes. However, an additional benefit of this approach is that the understanding of black holes would also elucidate the nature of gravity in the bulk, and not just on the brane. Furthermore, black holes may play an important role in both the astrophysics and cosmology of the braneworld. Although exact solutions have been found in lower dimensions [6,7], none are as yet known for black holes on 3-branes. Nevertheless, much information has been gleaned from appropriate limits [8–15]. In particular, Ref. [9] shows that the Schwarzschild metric is recovered on the brane to first nonlinear order, thus demonstrating the consistency of the braneworld with classical tests of general relativity.

In the Randall-Sundrum model, flatness of the 3-brane is maintained by fine-tuning the Randall-Sundrum brane tension. Subsequently, it was realized that “bent-braneworlds” (namely 3-branes with cosmological constant) may be obtained by relaxing the fine-tuning condition [16–20]. In Ref. [20], Karch and Randall investigated the nature of gravity in this class of bent braneworlds. They demonstrated that for both flat and de-Sitter 3-branes embedded in five-dimensional AdS, there is a single trapped massless graviton. However, for an AdS 3-brane embedded in AdS_5 (which is referred to as the AdS_{AdS} braneworld), a novel feature occurs — while a graviton is indeed bound to the brane, it turns out that the trapped graviton is *massive*, with mass [20]

$$M^2 \approx \frac{3L_5^2}{2L_4^4}. \quad (1.2)$$

This mass for the graviton has subsequently been obtained from a holographic interpretation of the Karch-Randall model, both from the bulk [21,22] and the dual theory on the brane [23,24]. For an alternative derivation see also [25]. The Newtonian law for a dS_4 brane embedded in a five-dimensional spacetime was investigated in [26]. Although theories with massive gravitons potentially suffer a van Dam-Veltman-Zakharov discontinuity [27,28], it was shown that this discontinuity is absent in the presence of a non-vanishing

cosmological constant [29,30]. While a graviton mass arising from explicit breaking of general covariance leads to inconsistencies at the quantum level [31], no such difficulties arise when the mass is generated dynamically, as in the holographic Karch-Randall model [23,24]. Nevertheless, many issues still arise in understanding models of massive gravity.

In this paper, we shall explore linearized gravity in the Karch-Randall model, and in particular we shall focus on the recovery of the Schwarzschild-AdS₄ geometry on the brane. In order to pursue this solution, we consider static radially-symmetric four-dimensional configurations, both in vacuum and with a point mass source on the brane. Our analysis of the off-brane profile of the gravitational field indicates the existence of a physical radion model in the Karch-Randall model. While this mode has previously been regarded as a gauge artifact [20], we show here that it cannot be completely gauged away, even when considering the effects of brane bending. We demonstrate that the effect of the radion shows up as a correction to the Schwarzschild-AdS₄ solution of comparable strength as from the standard tower of Kaluza-Klein modes.

We begin in section 2 by providing our bent braneworld conventions and setting up the general radially-symmetric metric ansatz. Within this ansatz, we obtain equations governing the Kaluza-Klein modes, as well as the radial wavefunctions on the brane. While much of the content here is standard, this provides the context on which the rest of the paper is based. In sections 3, we investigate the quasi-zero mode graviton, and in section 4 the Kaluza-Klein modes. Then in section 5 we demonstrate the presence of the Karch-Randall radion in the spectrum of transverse-traceless fluctuations. In section 6, we combine the graviton, radion and KK modes and obtain their coupling to a matter (point) source on the brane. Finally, we conclude in section 7.

2. Einstein Equations, Symmetries and Boundary Conditions.

The starting point for the Randall-Sundrum class of braneworlds is a warped product metric of the form

$$ds^2 = e^{2A(y)} \bar{g}_{\mu\nu}(x, y) dx^\mu dx^\nu + dy^2, \quad (2.1)$$

which is the most general metric with four-dimensional covariance. Here and throughout the paper, we adopt the convention that the Greek indices take values 0–3 and the Latin indices 0–4. Thus g_{mn} is the full five-dimensional metric, while $\bar{g}_{\mu\nu}$ is the four-dimensional metric “on the brane”.

For the case where $\bar{g}_{\mu\nu}(x)$ is independent of y , the five-dimensional Ricci tensor is simply expressed in terms of the four dimensional metric $\bar{g}_{\mu\nu}$ and warp factor $A(y)$ as

$$\begin{aligned} R_{\mu\nu} &= \bar{R}_{\mu\nu} - g_{\mu\nu}(4\dot{A}^2 + \ddot{A}), \\ R_{44} &= -4(\dot{A}^2 + \ddot{A}), \end{aligned} \quad (2.2)$$

where a dot indicates differentiation with respect to y . Away from the brane, the space is simply AdS_5 , so that the five dimensional metric g_{mn} satisfies the Einstein condition

$$R_{mn} = \Lambda_5 g_{mn}, \quad (2.3)$$

where Λ_5 is the bulk cosmological constant. Furthermore, for a bent braneworld, we assume the four dimensional metric $\bar{g}_{\mu\nu}$ similarly satisfies an Einstein condition

$$\bar{R}_{\mu\nu} = \Lambda_4 \bar{g}_{\mu\nu}, \quad (2.4)$$

this time with brane cosmological constant Λ_4 . As a result, the bulk Einstein equations take on the form

$$\begin{aligned} \Lambda_4 e^{-2A} - 4\dot{A}^2 - \ddot{A} &= \Lambda_5, \\ -4\dot{A}^2 - 4\ddot{A} &= \Lambda_5. \end{aligned} \quad (2.5)$$

Eliminating the second derivative (and hence removing sensitivity to any possible jump discontinuity in the warp factor), we obtain a simple equation

$$f^2 - \left(\frac{\dot{f}}{\kappa}\right)^2 = \left(\frac{k}{\kappa}\right)^2, \quad (2.6)$$

where $f = e^A$, and we have parameterized the bulk and brane cosmological constants in terms of the respective AdS radii¹

$$\Lambda_5 = -4\kappa^2 = -\frac{4}{L_5^2}, \quad \Lambda_4 = -3k^2 = -\frac{3}{L_4^2}. \quad (2.7)$$

Eq. (2.6) yields a simple solution for the warp factor

$$e^{2A} = \left(\frac{k}{\kappa}\right)^2 \cosh^2 \kappa(|y| - y_0), \quad (2.8)$$

where y_0 is a constant related to the value of the warp factor on the brane (which is located at $y = 0$). A natural choice is to demand $e^{2A} = 1$ on the brane, so that $\cosh \kappa y_0 = \kappa/k$. In the nearly-flat limit, this may be approximated as

$$e^{-2\kappa y_0} \approx \left(\frac{k}{2\kappa}\right)^2 \quad (k/\kappa \ll 1). \quad (2.9)$$

Note that we have inserted an absolute value in the y coordinate, which enforces the location of the brane. This discontinuity in the slope of the warp factor may be fixed by the Israel matching conditions accounting for the brane tension.

¹ Here we have assumed an AdS brane embedded in an AdS bulk, since this is what gives rise to a massive graviton. We will not elaborate on dS or flat braneworlds, but simply note that such solutions are obvious from Eq. (2.6) with slight modification of signs.

We treat the above solution as the braneworld vacuum, and consider the effect of an axially symmetric perturbation about this background. First note that the AdS_4 metric, $\bar{g}_{\mu\nu}$, may be written in global coordinates as

$$\bar{ds}^2 = -(1 + k^2 r^2) dt^2 + \frac{dr^2}{1 + k^2 r^2} + r^2 d\Omega^2. \quad (2.10)$$

With this parametrization in mind, we take the most general axially symmetric and static metric in $D = 4 + 1$ with the following form

$$ds^2 = e^{2A} [-e^{a+\bar{a}} dt^2 + e^{b-\bar{a}} dr^2 + e^c r^2 d\Omega_2^2] + dy^2, \quad (2.11)$$

where e^{2A} , given by Eq. (2.8), and $\bar{a} = \ln(1 + k^2 r^2)$ are fixed background quantities. This general ansatz depends on three functions a , b and c , all functions of r and y .

Substituting the metric (2.11) into the Einstein equation, (2.3), and keeping terms linear in a , b and c , we obtain the following equations

$$\begin{aligned} e^{\bar{a}} \left[a'' + \frac{2}{r} a' + \frac{1}{2} \bar{a}' (3a' - b' + 2c') \right] + e^{2A} \left[\ddot{a} + \dot{A} (5\dot{a} + \dot{b} + 2\dot{c}) \right] &= 6k^2 b + 16\pi G_5 e^{2A} (-T_t^t + \frac{1}{3} T) \delta(y), \\ e^{\bar{a}} \left[a'' + 2c'' - \frac{2}{r} (b' - 2c') + \frac{1}{2} \bar{a}' (3a' - b' + 2c') \right] + e^{2A} \left[\ddot{b} + \dot{A} (\dot{a} + 5\dot{b} + 2\dot{c}) \right] &= 6k^2 b + 16\pi G_5 e^{2A} (-T_r^r + \frac{1}{3} T) \delta(y), \\ e^{\bar{a}} \left[c'' + \frac{1}{r} (a' - b' + 4c') + \bar{a}' c' \right] + e^{2A} \left[\ddot{c} + \dot{A} (\dot{a} + \dot{b} + 6\dot{c}) \right] - \frac{2}{r^2} (b - c) &= 6k^2 b + 16\pi G_5 e^{2A} (-T_\theta^\theta + \frac{1}{3} T) \delta(y), \\ \ddot{a} + \ddot{b} + 2\ddot{c} + 2\dot{A} (\dot{a} + \dot{b} + 2\dot{c}) &= 16\pi G_5 (\frac{1}{3} T) \delta(y), \\ \dot{a}' + 2\dot{c}' - \frac{2}{r} (\dot{b} - \dot{c}) + \frac{1}{2} \bar{a}' (\dot{a} - \dot{b}) &= 0, \end{aligned} \quad (2.12)$$

where primes denote r derivatives. Note that we have included a brane stress energy tensor, $T_\nu^\mu \delta(y)$, as a source. This is in addition to the Karch-Randall brane tension itself, which is accounted for by the kink in the warp factor, (2.8). By integrating the equations (2.12) across the brane, one would obtain jump conditions on the metric functions a , b and c . Such conditions, in fact, are equivalent to the Israel matching conditions arising from the matter on the brane. These equations will be the starting point of our investigation.

We now discuss the gauge symmetries of these equations—more specifically the coordinate transformations that respect the axially symmetric, static form of the metric. In general, consider a coordinate transformation (gauge transformation) generated by functions v and u , such that

$$\begin{aligned} r &\rightarrow r + v(r, y), \\ y &\rightarrow y + u(r, y). \end{aligned} \quad (2.13)$$

By demanding that this particular coordinate transformation respects the form of the five dimensional metric, (2.11), we find that the functions v and u necessarily obey the following relations to linear order

$$e^{2A-\bar{a}}\dot{v} + u' = 0, \quad \dot{u} = 0. \quad (2.14)$$

Furthermore, the components of the metric transform under these residual coordinate transformations as follows

$$\begin{aligned} a(r, y) &\rightarrow a(r, y) + \bar{a}'v + 2\dot{A}u, \\ b(r, y) &\rightarrow b(r, y) - \bar{a}'v + 2\dot{A}u + 2v', \\ c(r, y) &\rightarrow c(r, y) + 2\dot{A}u + \frac{2}{r}v. \end{aligned} \quad (2.15)$$

The constraints, (2.14), may be integrated to provide the following form for the parameters of transformation

$$u = \chi(r), \quad v(r, y) = -e^{\bar{a}}\chi' \int^y dx e^{-2A(x)} + \phi(r). \quad (2.16)$$

Thus axially symmetric gauge transformations may be fully parameterized by two functions of r , namely $\chi(r)$ and $\phi(r)$. We will make use of such gauge transformations below, when working out the brane bending mode [2,3].

We now proceed to find solutions to the linearized equations of motion. We begin with the penultimate equation of (2.12), corresponding to R_{yy} , which may be integrated to give

$$a + b + 2c = \frac{8\pi G_5 T(r)}{3k^2} \dot{A}(y) \operatorname{sgn}(y) + f(r). \quad (2.17)$$

To arrive at this expression, we had to make use of the explicit form (2.8) for $A(y)$. Here $f(r)$ is an arbitrary function and $T(r)$ is the trace of the energy-momentum tensor on the brane. Since $\dot{A}(y) \operatorname{sgn}(y) \rightarrow 1$ as $y \rightarrow \infty$, we may satisfy the asymptotic conditions

$$\lim_{|y|, r \rightarrow \infty} a = \lim_{|y|, r \rightarrow \infty} b = \lim_{|y|, r \rightarrow \infty} c = 0, \quad (2.18)$$

provided $f(r) = -8\pi G_5 T(r)/3k^2$. Thus we may always set $a + b + 2c \approx 0$ away from the brane. However, in the presence of matter on the brane, the trace of the energy-momentum tensor provides an obstruction, and this will no longer be possible except in the asymptotic region.

We will return to this point when considering brane bending in the presence of sources on the brane. For now, we choose to examine radially symmetric excitations of the vacuum. Thus we set $T(r) = 0$, and in turn are allowed to simply choose $a + b + 2c = 0$. We refer to

this gauge choice as the traceless condition. By substituting $c = -\frac{1}{2}(a + b)$ into the final equation of (2.12) and integrating, we find

$$b' + \frac{1}{r}(a + 3b) - \frac{1}{2}\bar{a}'(a - b) = 0. \quad (2.19)$$

Furthermore, by taking into account that $\bar{a} = \ln(1 + k^2 r^2)$, we may eliminate a in terms of b

$$a = b - (1 + k^2 r^2)(rb' + 4b). \quad (2.20)$$

This allows us to rewrite the equations of motion in terms of the single function $b(r, y)$. We now separate variables for the transverse and traceless modes by decomposing $b(r, y) = \sum_{\mu} B(\mu) b(r|\mu) \psi(y|\mu)$. Separating variables on the second equation of (2.12) then results in a second order differential equation for the modes $b(r|\mu)$

$$b'' + \frac{4}{r} \frac{1 + 2k^2 r^2}{1 + k^2 r^2} b' + \frac{10k^2 - \mu^2}{1 + k^2 r^2} b = 0, \quad (2.21)$$

as well as an eigenvalue equation for the modes $\psi(y|\mu)$

$$e^{2A(y)}(\ddot{\psi} + 4\dot{A}\dot{\psi}) = -\mu^2 \psi. \quad (2.22)$$

From a four-dimensional point of view, $\psi(y|\mu)$ are the wavefunctions of the Kaluza-Klein gravitons, with Kaluza-Klein masses μ . Note that the wave equation on the brane, (2.21), may be written in the form

$$(\square_4 + 10k^2 - \mu^2)b + \frac{2}{r}(1 + 2k^2 r^2)b' = 0, \quad (2.23)$$

where \square_4 is the *scalar* Laplacian on the AdS_4 background. This is effectively the explicit form of the Lichnerowicz operator in our choice of coordinates with a spherically symmetric background.

Eq. (2.22) defines a Sturm-Liouville problem with e^{2A} as the measure of the normalization integral. Imposing the boundary conditions $e^{2A}\psi \rightarrow 0$ as $y \rightarrow \infty$ and $\dot{\psi} = 0$ at $y = 0$, we have a normalization integral

$$\int_0^\infty dy e^{2A(y)} \psi^2(y|\mu) = \psi(0|\mu) \frac{\partial \dot{\psi}(0|\mu)}{\partial \mu^2}. \quad (2.24)$$

In the following sections, we will examine the eigenvalue equation (2.22) for the quasi-zero mode, the radion and the Kaluza-Klein modes, as well as the scalar equation on the brane, (2.21).

3. The Karch-Randall Quasi-Zero Mode.

It was demonstrated in Ref. [20] that the graviton spectrum in the AdS braneworld is composed of a quasi-zero mode graviton trapped on the brane as well as a discrete tower of Kaluza-Klein modes. This spectrum may be determined by examination of the eigenvalue equation (2.22) for the graviton wavefunction. Here we shall examine the quasi-zero mode in detail and reproduce the expression, (1.2), for the graviton mass [20,21,32], by elementary means. The method that we employ does not depend on the details of the warp factor $A(y)$. The Kaluza-Klein modes will be considered in the following section. Note that we suppress the parameter μ whenever no possibility for confusion may arise.

We begin by assuming that the mode equation, (2.22), admits a normalized solution with eigenvalue $\mu^2 \ll m^2$. Here, m^2 is a scale introduced by the curvature of the brane [and may be taken to be k^2 for the Karch-Randall warp factor, (2.8)]. In this case, we can treat the right hand side of the equation perturbatively. To zeroth order, the right hand side of (2.22) is absent, and the equation possesses two solutions

$$u_1 = 1, \quad u_2 = \int^y dx e^{-4A(x)}, \quad (3.1)$$

with Wronskian $W(u_1, u_2) = e^{-4A}$. On writing the normalizable combination of the zeroth order solution as $\psi^{(0)}(y)$, the first order correction to the solution, $\psi^{(1)}(y)$, is determined by the inhomogeneous equation

$$\ddot{\psi}^{(1)} + 4\dot{A}\dot{\psi}^{(1)} = -\mu^2 e^{-2A} \psi^{(0)}. \quad (3.2)$$

This equation may be solved by the method of variation of coefficients; the solution is given by $\psi^{(1)} = \mu^2(c_1 u_1 + c_2 u_2)$, where c_1 and c_2 are both functions of y and are given by the expressions

$$\begin{aligned} c_1(y) &= \int^y dx \frac{e^{-2A} \psi^{(0)} u_2}{W(u_1, u_2)} = \int^y dx e^{2A} \psi^{(0)} u_2, \\ c_2(y) &= - \int^y dx \frac{e^{-2A} \psi^{(0)} u_1}{W(u_1, u_2)} = - \int^y dx e^{2A} \psi^{(0)} u_1. \end{aligned} \quad (3.3)$$

with the constants pertaining the indefinite integrals fixed by the normalizability condition. The eigenvalue μ^2 is determined by the Israel matching condition of the approximate solution $\psi = \psi^{(0)} + \psi^{(1)}$ on the brane, which for vanishing matter stress-energy reads simply

$$0 = \dot{\psi}(0) = \dot{\psi}^{(0)}(0) + \mu^2 c_2(0) \dot{u}_2(0). \quad (3.4)$$

This gives rise to the quasi-zero mode graviton mass

$$\mu^2 = - \frac{\dot{\psi}^{(0)}(0)}{c_2(0) \dot{u}_2(0)}, \quad (3.5)$$

which is valid provided $\mu^2 \ll m^2$.

In the case of the Karch-Randall brane model, the warp factor e^{2A} is given explicitly by (2.8). The zeroth order solutions to the mode equation, (2.22), reads simply

$$u_1 = 1, \quad u_2 = \tanh \eta (2 + \text{sech}^2 \eta), \quad (3.6)$$

where $\eta \equiv \kappa(y - y_0)$. As a result, the normalizable zeroth order solution is given by

$$\psi^{(0)} = 1 - \frac{1}{2} \tanh \eta (2 + \text{sech}^2 \eta). \quad (3.7)$$

Proceeding to the first order correction, we compute c_1 and c_2 , which are given by the integrals

$$c_1 = -\frac{1}{3k^2} \int_{\eta}^{\infty} d\zeta \psi^{(0)}(\zeta) (2 \cosh^2 \zeta + 1) \tanh \zeta, \quad c_2 = \frac{1}{3k^2} \int_{\eta}^{\infty} d\zeta \psi^{(0)}(\zeta) \cosh^2 \zeta. \quad (3.8)$$

Carrying out the integrations, we find the quasi-zero mode wavefunction is given to first order in $\frac{\mu^2}{\kappa^2}$ as

$$\begin{aligned} \psi &= \psi^{(0)} + \mu^2 (c_1 u_1 + c_2 u_2) \\ &= 1 - \frac{1}{2} \tanh \eta (2 + \text{sech}^2 \eta) \\ &\quad + \frac{\mu^2}{3k^2} \left[\frac{1}{2} \text{sech}^2 \eta - 2(1 - \tanh^2 \eta) + \frac{1}{6} \left(1 - \frac{1}{2} \tanh \eta (2 + \text{sech}^2 \eta) \right) \right. \\ &\quad \left. + \ln(1 + e^{-2\eta}) \left(1 + \frac{1}{2} \tanh \eta (2 + \text{sech}^2 \eta) \right) \right]. \end{aligned} \quad (3.9)$$

The four-dimensional mass parameter μ is calculated according to (3.5); we find

$$\mu^2 \approx 6k^2 e^{-2\kappa y_0} = \frac{3}{2} \frac{k^4}{\kappa^2}, \quad (3.10)$$

provided $\mu^2/k^2 \ll 1$. The final relation was obtained using the relation (2.9), and reproduces the expected value of the Karch-Randall graviton mass, (1.2).

Having identified the quasi-zero mode wavefunction, (3.9), and the graviton mass, (3.10), we now proceed to determine the profile of $b(r)$ on the brane. This is accomplished by finding solutions of the AdS_4 wave equation, (2.21), consistent with boundary conditions. By introducing a new variable $\xi = -k^2 r^2$, and defining $b = f/(1 - \xi)$, Eq. (2.21) may be transformed to the form

$$\frac{\partial^2}{\partial \xi^2} f + \frac{5}{2\xi} \frac{\partial}{\partial \xi} f = -\frac{\bar{\epsilon}}{\xi(1 - \xi)} f, \quad (3.11)$$

where $\bar{\epsilon} = \mu^2/4k^2$. While this is easily solved in terms of hypergeometric functions, for the quasi-zero mode, $\bar{\epsilon} \ll 1$, we find it instructive to examine the series solution using the same perturbative approach developed above.

To zeroth order, there are simply two linearly independent solutions

$$u_1 = 1, \quad u_2 = (-\xi)^{-\frac{3}{2}}. \quad (3.12)$$

Imposing boundary conditions at spatial infinity, we demand for the zeroth order solution that $b^{(0)} \rightarrow 0$ as $\xi \rightarrow -\infty$. This ensures that the spacetime remains asymptotically AdS_4 on the brane. This requirement selects the second solution, so that we take

$$f^{(0)} = C(-\xi)^{-\frac{3}{2}} = \frac{C}{k^3 r^3}, \quad (3.13)$$

where C is a constant. The next order solution can be written as

$$f^{(1)} = -\frac{4\bar{\epsilon}C}{3} \left[(-\xi)^{-1/2} - \tan^{-1}(-\xi)^{-1/2} + \frac{1}{2}(-\xi)^{-3/2} \ln(1 - \xi) \right]. \quad (3.14)$$

As a result, $b(r)$ is given up to first order in $\bar{\epsilon} \approx 3k^2/8\kappa^2$ by

$$b(r) = \frac{C}{1 + k^2 r^2} \left[\frac{1}{k^3 r^3} \left(1 - \frac{k^2}{4\kappa^2} \ln(1 + k^2 r^2) \right) - \frac{k^2}{2\kappa^2} \left(\frac{1}{kr} - \tan^{-1} \frac{1}{kr} \right) \right]. \quad (3.15)$$

The remaining functions, $a(r)$ and $c(r)$ may now be determined through Eq. (2.20) and the relation $c = -\frac{1}{2}(a + b)$. The result is

$$\begin{aligned} a(r) &= \frac{C}{1 + k^2 r^2} \left[\frac{1}{kr} \left(1 - \frac{k^2}{4\kappa^2} \ln(1 + k^2 r^2) \right) + \frac{k^2}{2\kappa^2} (3 + 2k^2 r^2) \left(\frac{1}{kr} - \tan^{-1} \frac{1}{kr} \right) \right], \\ c(r) &= -\frac{C}{2} \left[\frac{1}{k^3 r^3} \left(1 - \frac{k^2}{4\kappa^2} \ln(1 + k^2 r^2) \right) + \frac{k^2}{\kappa^2} \left(\frac{1}{kr} - \tan^{-1} \frac{1}{kr} \right) \right]. \end{aligned} \quad (3.16)$$

Given the quasi-zero mode graviton, this above solution ought to reproduce the Schwarzschild-AdS black hole at linearized order. However, in order to compare this metric, computed in the transverse-traceless gauge, with that in the standard Schwarzschild-AdS form, we must transform the metric component c to zero. This may be accomplished by making use of the residual coordinate transformations, (2.15) and (2.16), with parameter $\phi(r)$

$$a \rightarrow a + \frac{2k^2 r}{1 + k^2 r^2} \phi(r), \quad b \rightarrow b - \frac{2k^2 r}{1 + k^2 r^2} \phi(r) + 2\phi'(r), \quad c \rightarrow c + \frac{2}{r} \phi(r). \quad (3.17)$$

We see that c may be eliminated by choosing $\phi(r) = -\frac{r}{2}c(r)$, whereupon the transformed metric components take the form

$$\begin{aligned} a(r) &= -\frac{2G_4M}{r(1+k^2r^2)} \left[1 - \frac{k^2}{4\kappa^2} \ln(1+k^2r^2) + \frac{k^3r}{\kappa^2}(1+k^2r^2) \left(\frac{1}{kr} - \tan^{-1} \frac{1}{kr} \right) \right], \\ b(r) &= \frac{2G_4M}{r(1+k^2r^2)} \left[1 + \frac{k^2}{2\kappa^2} - \frac{k^2}{4\kappa^2} \ln(1+k^2r^2) \right]. \end{aligned} \quad (3.18)$$

Here we have identified the dimensionless constant C with the mass according to $C = -4G_4Mk/3$. This ensures that, up to terms of $\mathcal{O}(k^2/\kappa^2)$, this solution reproduces the linearized portion of Schwarzschild-AdS

$$ds^2 = - \left(1 - \frac{2G_4M}{r} + k^2r^2 \right) dt^2 + \left(1 - \frac{2G_4M}{r} + k^2r^2 \right)^{-1} dr^2 + r^2 d\Omega^2. \quad (3.19)$$

The normalization of C will be obtained in section 6, when we consider an explicit source on the brane.

This demonstrates that the quasi-zero mode graviton is responsible for the long-range gravitational interaction on the brane, even though it has mass $\mu^2 \sim k^4/\kappa^2$. This is a feature of the AdS_4 geometry, in that, for the limit we are considering, the mass is infinitesimal compared to the natural scale of the AdS curvature (*i.e.* $\bar{\epsilon} \ll 1$ for the dimensionless mass $\bar{\epsilon}$). Because of this, the background curvature provides a cutoff to the space before any effects of the graviton mass may be discerned. Furthermore, we see that (at least to this order) no van Dam-Veltman-Zakharov discontinuity [27,28] arises, which is consistent with the results of Refs. [29,30].

4. Kaluza-Klein Modes.

Having examined the quasi-zero mode in detail, we now turn to the massive Kaluza-Klein spectrum. Although a perturbative investigation is no longer valid in this regime, we find that the wave equation for the Kaluza-Klein modes, (2.22), admits solutions of hypergeometric form. To see this, we rewrite the wavefunction $\psi(y)$ according to $\psi(y) = \phi(y)\text{sech}^2\kappa(y-y_0)$ and introduce the variable $x = \tanh\kappa(y-y_0)$. As a result, the mode equation, (2.22), is converted to an associated Legendre equation

$$(1-x^2)\frac{d^2\phi}{dx^2} - 2x\frac{d\phi}{dx} + \left[l(l+1) - \frac{m^2}{1-x^2} \right] \phi = 0, \quad (4.1)$$

with $m = 2$ and $l = E_0 - 2$. Here, E_0 is the lowest energy eigenvalue for the massive graviton representation $D(E_0, s=2)$ of the AdS_4 isometry group $\text{SO}(2,3)$, and is given by

$$E_0 = \frac{3}{2} + \frac{1}{2}\sqrt{9 + \frac{4\mu^2}{k^2}}. \quad (4.2)$$

The range of the new variable x is $-(1 - k^2/\kappa^2)^{1/2} < x < 1$, with the AdS_5 boundary located at $x = 1$ and the brane located at $x = -(1 - k^2/\kappa^2)^{1/2} \equiv \bar{x}$. The normalization integral of $\psi(y)$ over y , (2.24), becomes that of $\phi(x)$ over x with a constant measure.

The general solution to the associated Legendre equation, (4.1), is a linear combination of $P_l^m(x)$ and $Q_l^m(x)$. Only $P_l^m(x)$ is retained since the other solution is singular at $x = 1$, which causes the normalization integral to diverge. Therefore the Kaluza-Klein graviton wavefunctions are of the form

$$\psi(x) = C(1 - x^2)P_l^m(x), \quad (4.3)$$

where the normalization constant may be determined by (2.24). The quantization condition on l arises from the Israel matching condition on the brane, which is simply $d\psi/dx|_{x=\bar{x}} = 0$ in the absence of matter on the brane. We now consider two cases, the first in the formal absence of a brane (vanishing brane tension), and the second with a brane.

4.1 Without a brane

At zero tension, $y_0 = 0$, and the AdS_4 curvature is identified with that of AdS_5 , $k = \kappa$. This provides an interval $0 < x < 1$ to the right of the brane. However, for $y_0 = 0$, the kink at the brane is absent, and we may drop the absolute value on y in the warp factor. As a result, without a brane the variable x extends through the full interval of $[-1, 1]$. It follows from the theory of spherical harmonics that normalizability restricts l to positive integers, $l \geq |m|$. In this case, this indicates that $l = 2, 3, 4, \dots$, or $E_0 = 4, 5, 6, \dots$

Inverting the spin-2 relation (4.2) for the graviton mass, we find the zero tension spectrum

$$\mu^2 = E_0(E_0 - 3)k^2, \quad E_0 = 4, 5, 6, \dots, \quad (4.4)$$

a result obtained previously by Karch and Randall [20] and which agrees with the embedding of AdS_4 in AdS_5 . The normalizable wave function reads

$$\psi_{E_0}(x) = \sqrt{\frac{(2E_0 - 3)(E_0 - 4)!}{2E_0!}} \frac{\kappa^{3/2}}{k^2} (1 - x^2) P_{E_0-2}^2(x). \quad (4.5)$$

In this case there is no massless four-dimensional graviton, which is understood since there is in fact no brane present.

4.2 With a brane

With the introduction of a brane, we now work on the right side of the brane, $\bar{x} < x < 1$, and impose the Israel matching condition $d\psi/dx|_{x=\bar{x}} = 0$. The requirement that the solution satisfies this condition now renders the l (or equivalently E_0) non-integers.

Nevertheless, we retain the standard AdS₄ expression (4.4) for the mass with a non-integer $E_0 = l + 2$.

For non-integral l , it is more convenient to work with the representation of the associated Legendre functions in terms of hypergeometric functions. In particular,

$$P_{E_0-2}^2(x) = \frac{\Gamma(E_0 + 1)}{8\Gamma(E_0 - 3)}(1 - x^2) {}_2F_1(4 - E_0, E_0 + 1; 3; \frac{1 - x}{2}). \quad (4.6)$$

We wish to investigate the Kaluza-Klein spectrum in the nearly flat brane limit, corresponding to $k \ll \kappa$, or $\bar{x} \approx -1 + k^2/2\kappa^2$. Thus we need to examine the behavior of ψ near $x = -1$. Since this corresponds to the argument of ${}_2F_1$ approaching one, we invoke the transformation theory of hypergeometric functions. Some of the relevant properties are given in Appendix A.

Low lying modes

The low lying states are those with $\mu \ll \kappa$, for which the quantum number E_0 remains close to an integer. By writing $\epsilon = (1 + x)/2$, which is small near the brane, the function ψ takes the asymptotic form

$$\begin{aligned} \psi \simeq C \left\{ -\frac{4}{\pi}[1 + E_0(E_0 - 3)\epsilon] \sin E_0\pi + 2\frac{\Gamma(E_0 + 1)}{\Gamma(E_0 - 3)}\epsilon^2 \left[\cos E_0\pi \right. \right. \\ \left. \left. + \frac{1}{\pi} \left(-\frac{3E_0 + 1}{2(E_0 - 1)} + 2\gamma + 2\psi(E_0) + \ln \epsilon \right) \sin E_0\pi \right] + O(\epsilon^3) \right\}. \end{aligned} \quad (4.7)$$

The matching condition then yields for E_0

$$E_0 = n + 3 + (n + 1)(n + 2)\frac{k^2}{4\kappa^2}, \quad (4.8)$$

where n is a non-negative integer. Note that $n = 0$ yields $E_0 = 3 + k^2/2\kappa^2$, which corresponds to the quasi-zero mode. In particular, using (4.4), we see that this mode has mass $\mu^2 = 3k^4/2\kappa^2$, which agrees with what was found in the previous section. Except for the quasi-zero mode, the wave function (4.5) with integer E_0 remains a good approximation in the bulk.

Higher modes

For the higher eigenvalues, the omitted terms in the expansion of the hypergeometric function must to be restored, and we need to evaluate the confluent limit of a hypergeometric function with a fine-tuned small argument and large parameters. In terms of

$x = -1 + \zeta^2/2n^2$ with $\zeta = \mathcal{O}(1)$ and $n \gg 1$, we have $\zeta = (\mu/\kappa)e^{\kappa y}$ and the wavefunction becomes

$$\psi = \frac{\kappa^2}{k^2} \sqrt{k} \zeta^2 [J_2(\zeta) \cos n\pi + N_2(\zeta) \sin n\pi], \quad (4.9)$$

where $J_2(\zeta)$ and $N_2(\zeta)$ are the second Bessel and Newmann functions, respectively. The derivation of this formula appears in Appendix A. The quantization condition becomes

$$\tan n\pi = -\frac{J_1(\frac{\mu}{\kappa})}{N_1(\frac{\mu}{\kappa})}. \quad (4.10)$$

This limit reproduces the Randall-Sundrum model with a flat brane.

5. Brane Bending and the Radion Mode.

In the previous two sections, we have demonstrated that the spectrum of transverse-traceless gravitational modes is composed of a quasi-zero mode graviton (trapped on the brane) as well as a discrete Kaluza-Klein tower. The mode expansion may be given in terms of the four-dimensional E_0 eigenvalue (equivalently the KK mass) and the eigenfunctions along the y direction, $\psi(y)$. It is important to realize that this analysis was performed under the requirement of two boundary conditions. Firstly, normalizability of $\psi(y)$ demands that it vanishes sufficiently fast at the AdS_5 boundary, $y \rightarrow \infty$. And, secondly, the Israel matching condition in the absence of matter imposes the vanishing of the derivative at $y = 0$ (*i.e.* on the brane), namely $\dot{\psi}(0) = 0$. Consequently, values of E_0 less than two were immediately ruled out. This follows from the asymptotic expression of $\psi(y)$ as $y \rightarrow 0$, since for $E_0 < 2$ it is no longer possible to satisfy the Israel matching condition on (4.3) without introducing the companion solution, which is not normalizable.

However, one important exception must be made to the above boundary conditions. Namely, a solution which blows up would in fact be acceptable, provided the non-normalizable piece can be removed by an appropriate residual gauge transformation, (2.15). These gauge transformations connect the behavior of $\psi(y)$ near the brane to the one near the boundary of AdS_5 ; the wave function obtained in this manner satisfies both boundary conditions at the expense of moving the brane away from $y = 0$ in the new coordinates. Because of this, the transformation is commonly denoted as brane bending [2,3].

Inserting the parameters (2.16) into the residual transformation (2.15), we find a general parametrization for the gauge transformations

$$\begin{aligned} \delta a &= -\frac{1}{k^2} (e^{\bar{a}} \bar{a}' \chi' - 2k^2 \chi) \dot{A} + \bar{a}' \phi, \\ \delta b &= -\frac{1}{k^2} (2e^{\bar{a}} \chi'' + e^{\bar{a}} \bar{a}' \chi' - 2k^2 \chi) \dot{A} - \bar{a}' \phi + 2\phi', \\ \delta c &= -\frac{1}{k^2} \left(\frac{2}{r} e^{\bar{a}} \chi' - 2k^2 \chi \right) \dot{A} + \frac{2}{r} \phi, \end{aligned} \quad (5.1)$$

where χ and ϕ are both functions of r only. Here it is apparent that ϕ generates four-dimensional gauge transformations on the brane as in (3.17), while χ is responsible for transformations involving the bulk. For the latter, the y -profile is given simply by \dot{A} . It is this (and only this) form of non-normalizable behavior that may be canceled through brane bending. Since $\psi \sim \dot{A}$ solves the eigenmode equation (2.22) with $\mu^2 = -2k^2$, extra care must be taken in analyzing this case. This provides the origin of the Karch-Randall radion, which we examine here.

For $\mu^2 = -2k^2$, corresponding to $E_0 = 2$, the mode equation (2.22) admits two linearly independent solutions,

$$\psi_1 = \dot{A}, \quad (5.2)$$

as indicated above, and

$$\psi_2 = (\kappa - \dot{A})^2, \quad (5.3)$$

which can be verified by direct substitution. The ψ_1 mode, which does not vanish at the AdS_5 boundary, has the same y -dependence as the brane bending term of (5.1). On the other hand, ψ_2 is normalizable, and cannot be removed by gauge transformations. The appropriate linear combination of ψ_1 and ψ_2 that solves the Israel matching condition is given by

$$\psi_{\text{radion}} = \psi_1 + \frac{1}{2}(\kappa - \dot{A}(0))^{-1}\psi_2 = \dot{A} + \frac{1}{2} \frac{(\kappa - \dot{A})^2}{\kappa - \dot{A}(0)}. \quad (5.4)$$

We identify this as the radion mode.

Since the radion corresponds to $E_0 = 2$, the radial function $b(r)$ is given by solutions to (2.21) with $\mu^2 = -2k^2$. However, instead of solving (2.21) directly, we note that there is already a natural parametrization for $b(r)$, given by the gauge transformation (5.1) itself. This suggests that we choose an ansatz for the radion mode according to

$$\begin{aligned} a_{\text{radion}}(r, y) &\equiv a_{BR}(r)\psi_{\text{radion}}(y) = -\frac{1}{k^2}(e^{\bar{a}}\bar{a}'\chi' - 2k^2\chi)\psi_{\text{radion}}, \\ b_{\text{radion}}(r, y) &\equiv b_{BR}(r)\psi_{\text{radion}}(y) = -\frac{1}{k^2}(2e^{\bar{a}}\chi'' + e^{\bar{a}}\bar{a}'\chi' - 2k^2\chi)\psi_{\text{radion}}, \\ c_{\text{radion}}(r, y) &\equiv c_{BR}(r)\psi_{\text{radion}}(y) = -\frac{1}{k^2}\left(\frac{2}{r}e^{\bar{a}}\chi' - 2k^2\chi\right)\psi_{\text{radion}}. \end{aligned} \quad (5.5)$$

The transverse-traceless condition requires χ to satisfy the homogeneous equation,

$$\chi'' + \left(\frac{2}{r} + \bar{a}'\right)\chi' - 4k^2e^{-\bar{a}}\chi = 0, \quad (5.6)$$

which is simply the scalar equation

$$[-\square + 4k^2]\chi = 0, \quad (5.7)$$

in the AdS_4 background. Since the equation for a massive scalar in AdS_4 has the form $[-\square + E_0(E_0 - 3)]\varphi = 0$, this identifies the radion (as well as the corresponding brane bending) mode as an $E_0 = 4$ scalar [20]. Provided this scalar equation is solved, it is easy to verify that b_{BR} solves the spin-2 equation (2.21) with $\mu^2 = -2k^2$. This formally corresponds to a spin-2 value of $E_0 = 2$, but, as indicated by our choice of χ , more properly should be realized as a spin-0 value of $E_0 = 4$.

The homogeneous equation (5.6) can be transformed to hypergeometric type by introducing the variable $\xi = -k^2 r^2$

$$\xi(1 - \xi)\frac{d^2\chi}{d\xi^2} + \left(\frac{3}{2} - \frac{5}{2}\xi\right)\frac{d\chi}{d\xi} + \chi = 0. \quad (5.8)$$

The solution can be written as

$$\chi(r) = C_1 {}_2F_1\left(2, -\frac{1}{2}; \frac{3}{2}; -k^2 r^2\right) + \frac{C_2}{r} {}_2F_1\left(\frac{3}{2}, -1; \frac{1}{2}; -k^2 r^2\right), \quad (5.9)$$

with

$$\begin{aligned} {}_2F_1\left(2, -\frac{1}{2}; \frac{3}{2}; -k^2 r^2\right) &= \frac{3}{4} + \frac{1}{4}(1 + 3k^2 r^2)\frac{\tan^{-1} kr}{kr}, \\ {}_2F_1\left(\frac{3}{2}, -1; \frac{1}{2}; -k^2 r^2\right) &= 1 + 3k^2 r^2. \end{aligned} \quad (5.10)$$

The combination that vanishes as $r \rightarrow \infty$ reads

$$\chi(r) = C \left[\frac{1 + 3k^2 r^2}{3kr} \tan^{-1} \frac{1}{kr} - 1 \right], \quad (5.11)$$

Substituting (5.11) to (5.5), we find

$$\begin{aligned} a_{BR}(r) &= \frac{4}{3}C \left[\frac{1}{kr} \tan^{-1} \frac{1}{kr} - \frac{1}{1 + k^2 r^2} \right], \\ b_{BR}(r) &= \frac{4}{3}C \frac{1}{k^2 r^2} \left[-\frac{1}{kr} \tan^{-1} \frac{1}{kr} - \frac{1}{1 + k^2 r^2} \right], \\ c_{BR}(r) &= \frac{2}{3}C \frac{1}{k^2 r^2} \left[\frac{1 - k^2 r^2}{kr} \tan^{-1} \frac{1}{kr} + 1 \right]. \end{aligned} \quad (5.12)$$

As will be seen in the following section, the radion mode is sourced by the trace of the energy-momentum tensor on the brane. In particular, it will contribute to the static metric with a point source. Asymptotic expansion of (5.12) indicates that $b_{\text{radion}}(r, y) = \mathcal{O}(1/r^4)$ for large r , which dominates over the quasi-zero mode graviton contribution to $b(r, y)$. This presents a challenge to the recovery of the Schwarzschild- AdS_4 metric on the brane.

However, performing a transformation on the brane to bring the metric into a standard form, $c_{\text{radion}}(r, 0) = 0$, we obtain

$$\begin{aligned} a_{\text{radion}}(r, 0) &= \frac{2C}{3(1+k^2r^2)} \left(\frac{1+3k^2r^2}{kr} \tan^{-1} \frac{1}{kr} - 3 \right) \psi_{\text{radion}}(0), \\ b_{\text{radion}}(r, 0) &= \frac{2C}{3(1+k^2r^2)} \left(\frac{1-k^2r^2}{kr} \tan^{-1} \frac{1}{kr} + 1 \right) \psi_{\text{radion}}(0), \end{aligned} \quad (5.13)$$

which now becomes subleading relative to the quasi-zero mode graviton. While it would be important to determine if the radion mode remains subleading beyond the linearized approximation, this is outside the scope of our present analysis.

6. Matter Sources on the Brane.

Having discussed the general features of the braneworld linearized gravity system in the transverse traceless gauge, we now return to the issue of matter sources on the brane. Recall from Eq. (2.17) that the trace of the stress-energy tensor on the brane provides an obstruction to imposing the traceless condition, $c = -\frac{1}{2}(a+b)$. In particular, for a static mass point on the brane, where $T_{00} = M\delta^3(\vec{r})$, the constraint, (2.17), takes the form

$$a + b + 2c = -\frac{8\pi G_5 M}{3k^2} \dot{A}(y) \delta^3(\vec{r}) + f(r), \quad (6.1)$$

where $f(r)$ is an arbitrary function of r and we restrict our attention to $y \geq 0$. Although the source is located on the brane, the delta function survives as we move into the bulk.

Although this effect presents an obstruction to working in transverse-traceless gauge, the y -profile of (6.1), given by \dot{A} , was identified in the previous section as that related to brane bending and the radion. In particular, Eq. (6.1) indicates that the mass (or trace of the stress tensor on the brane) provides a source for the ψ_1 mode of the previous section. If inducing a non-trivial ψ_1 where the only effect of a source on the brane, then there would be no additional physical consequences, as this is purely gauge. However, as indicated by (5.4), ψ_1 fails to satisfy the Israel matching condition by itself, and necessarily comes in through the radion combination. This demonstrates that the radion, in fact, is being sourced by matter on the brane.

To provide a heuristic accounting for the radion, consider writing the metric functions as

$$b(r, y) = \tilde{b}_{TT}(r, y) + b_{\text{radion}}(r, y) = \tilde{b}_{TT}(r, y) + b_{BR}(r) \psi_{\text{radion}}(y) \quad (6.2)$$

(and similarly for a and c). Here, \tilde{b}_{TT} corresponds to the $E_0 > 2$ solutions of section 3 (quasi-zero mode graviton) and section 4 (KK gravitons) in the transverse-traceless gauge. Similarly, b_{radion} is given as in section 6, with the exception that it is no longer a traceless

mode, but is sourced by (6.1). As a slight complication, ψ_{radion} contains both ψ_1 and ψ_2 , with only ψ_1 sourced by (6.1), and the ψ_2 mode remaining traceless. For this reason, it is more convenient, in fact, to split off ψ_2 , and incorporate it into \tilde{b}_{TT} . Thus we write instead

$$b(r, y) = b_{TT}(r, y) + b_{BR}(r)\psi_1(y) \quad (6.3)$$

(and again similarly for a and c), where b_{TT} , while remaining transverse-traceless, now has an $E_0 = 2$ component as well as the usual $E_0 > 2$ contributions.

By substituting Eq. (6.3) into (6.1), we find that the scalar radion mode χ of (5.6) now satisfies an inhomogeneous equation with source

$$\chi'' + \left(\frac{2}{r} + \bar{a}'\right)\chi' - 4k^2 e^{-\bar{a}}\chi = \frac{4\pi G_5 M}{3}\delta^3(\vec{r}). \quad (6.4)$$

This essentially fixes the constant of (5.11) to be

$$C = -\frac{2G_5 M k}{\pi}. \quad (6.5)$$

Consequently, the complete normalized radion mode b_{radion} is expected to be

$$\begin{aligned} b_{\text{radion}}(r, y) &= b_{BR}(r)\psi_{\text{radion}}(y) \\ &= \frac{8G_5 M}{3\pi} \frac{1}{kr^2} \left(\frac{1}{kr} \tan^{-1} \frac{1}{kr} + \frac{1}{1 + k^2 r^2} \right) \left(\dot{A} + \frac{1}{2} \frac{(\kappa - \dot{A})^2}{\kappa - \dot{A}(0)} \right), \end{aligned} \quad (6.6)$$

with the ψ_1 term arising from χ and the ψ_2 term arising from b_{TT} . Only the former may be removed by a brane bending gauge transformation, leaving the component proportional to ψ_2 as the physical radion mode.

The determination of a_{TT} , b_{TT} and c_{TT} of the solution (6.3) involves the inversion of the Lichnerowicz operator with a AdS_4 background, which can be made explicit for this special case. As is always the case for an elliptic equation of two variables, the solution can be written either as a sum over the modes (obtained in the previous sections) which are oscillatory in y or as a sum over modes which are oscillatory in r . The former approach highlights the asymptotic AdS_4 behavior on the brane for $r \rightarrow \infty$ but overshadows the damping behavior off the brane, especially for higher modes. For a source only on the brane, the off-brane damping (which is implicit in the summation over the KK modes) is an important feature of the solution. To make this more explicit, we now switch to the latter (oscillatory in r) approach in constructing integral representations of a_{TT} , b_{TT} and c_{TT} .

On writing $\mu^2/k^2 = -\frac{9}{4} - \nu^2$, which corresponds to complex $E_0 = \frac{3}{2} + i\nu$, equation (2.21) defines a Sturm-Liouville problem with a continuous spectrum ν^2 and measure $r^4(1 + k^2 r^2)$ for the normalization integral. The radial eigenfunctions read

$$b_\nu(r) \equiv b(r|\mu)|_{\mu^2/k^2 = -9/4 - \nu^2} = {}_2F_1\left(\frac{7}{4} + \frac{i}{2}\nu, \frac{7}{4} - \frac{i}{2}\nu; \frac{5}{2}; -k^2 r^2\right), \quad (6.7)$$

and display the asymptotic behavior

$$b_\nu(r) \simeq \sqrt{2}C_\nu^{-1} \frac{\sin(\nu \ln kr + \delta_\nu)}{(kr)^{\frac{7}{2}}}, \quad (6.8)$$

for large kr . The phase shift δ_ν is given by

$$e^{2i\delta_\nu} = -\frac{\Gamma(i\nu)\Gamma(\frac{3}{4} - \frac{i}{2}\nu)\Gamma(\frac{7}{4} - \frac{i}{2}\nu)}{\Gamma(-i\nu)\Gamma(\frac{3}{4} + \frac{i}{2}\nu)\Gamma(\frac{7}{4} + \frac{i}{2}\nu)}, \quad (6.9)$$

while the normalization constant is

$$C_\nu^2 = \frac{2}{9\pi^2}(\nu^2 + \frac{9}{4})\Gamma^2(\frac{3}{4} + \frac{i}{2}\nu)\Gamma^2(\frac{3}{4} - \frac{i}{2}\nu)\nu \sinh \pi\nu. \quad (6.10)$$

The corresponding solution of (2.22) that vanishes on the AdS_5 boundary is

$$\begin{aligned} \psi_\nu(y) &\equiv \psi(y|\mu)\big|_{\mu^2/k^2 = -9/4 - \nu^2} \\ &= \text{sech}^4 \kappa(y - y_0) {}_2F_1\left(\frac{5}{2} + i\nu, \frac{5}{2} - i\nu; 3; \frac{1 - \tanh \kappa(y - y_0)}{2}\right). \end{aligned} \quad (6.11)$$

This allows us to decompose the transverse-traceless function b_{TT} in terms of a spectral density β_ν as

$$\begin{aligned} b_{TT}(r, y) &= \int_0^\infty \frac{d\nu}{\pi} \beta_\nu b_\nu(r) \frac{\psi_\nu(y)}{\psi_\nu(0)} \\ &= \int_0^\infty \frac{d\nu}{\pi} \beta_\nu {}_2F_1\left(\frac{7}{4} + \frac{i}{2}\nu, \frac{7}{4} - \frac{i}{2}\nu, \frac{5}{2}; -k^2 r^2\right) \frac{\psi_\nu(y)}{\psi_\nu(0)}. \end{aligned} \quad (6.12)$$

In order to determine β_ν , we apply the Israel matching conditions, $\dot{b}(r, y) = 0$ to (6.3). The result is simply $\dot{b}_{TT}(r, 0) = -b_{BR}(r)\ddot{A}(0)$, which reads more explicitly

$$\int_0^\infty \frac{d\nu}{\pi} \beta_\nu {}_2F_1\left(\frac{7}{4} + \frac{i}{2}\nu, \frac{7}{4} - \frac{i}{2}\nu, \frac{5}{2}; -k^2 r^2\right) = -\frac{8G_5 M}{3\pi} \frac{1}{kr^2} \left(\frac{1}{kr} \tan^{-1} \frac{1}{kr} + \frac{1}{1 + k^2 r^2} \right) \ddot{A}(0). \quad (6.13)$$

Notice that the $\delta^3(\vec{r})$ function pertaining to the source is weak with respect to the measure $r^4(1 + k^2 r^2)$ and subsequently is omitted from the matching condition. We now invert this integral to solve for β_ν . To do so, we use the result of Appendix B, which demonstrates that any square integrable function of r with respect to the measure $r^4(1 + k^2 r^2)$ can be expressed as a generalized Fourier integral

$$f(r) = \int_0^\infty \frac{d\nu}{\pi} \mathcal{F}(\nu) {}_2F_1\left(\frac{7}{4} + \frac{i}{2}\nu, \frac{7}{4} - \frac{i}{2}\nu, \frac{5}{2}; -k^2 r^2\right), \quad (6.14)$$

with \mathcal{F} given by the inverse transformation

$$\mathcal{F}(\nu) = C_\nu^2 k^5 \int_0^\infty dr r^4 (1 + k^2 r^2) f(r) {}_2F_1\left(\frac{7}{4} + \frac{i}{2}\nu, \frac{7}{4} - \frac{i}{2}\nu; \frac{5}{2}; -k^2 r^2\right). \quad (6.15)$$

It thus follows that

$$\beta_\nu = -\frac{4G_5 M k}{3\pi} \ddot{A}(0) C_\nu^2 \int_0^\infty dx \left[(1+x) \tan^{-1} \frac{1}{\sqrt{x}} + \sqrt{x} \right] {}_2F_1\left(\frac{7}{4} + \frac{i}{2}\nu, \frac{7}{4} - \frac{i}{2}\nu; \frac{5}{2}; -x\right). \quad (6.16)$$

Using the identity [33]

$$(1+x) {}_2F_1\left(\frac{7}{4} + \frac{i}{2}\nu, \frac{7}{4} - \frac{i}{2}\nu; \frac{5}{2}; -x\right) = -\frac{6}{\nu^2 + \frac{1}{4}} \frac{d}{dx} \left[(1+x)^2 {}_2F_1\left(\frac{7}{4} + \frac{i}{2}\nu, \frac{7}{4} - \frac{i}{2}\nu; \frac{3}{2}; -x\right) \right], \quad (6.17)$$

and integrating by parts, we find that

$$\begin{aligned} \beta_\nu = -4G_5 M k \ddot{A}(0) \frac{C_\nu^2}{\nu^2 + \frac{1}{4}} - \frac{4G_5 M k}{3\pi} \ddot{A}(0) C_\nu^2 \int_0^\infty \frac{dx}{\sqrt{x}} \left[x {}_2F_1\left(\frac{7}{4} + \frac{i}{2}\nu, \frac{7}{4} - \frac{i}{2}\nu; \frac{5}{2}; -x\right) \right. \\ \left. - 3 \frac{1+x}{\nu^2 + \frac{1}{4}} {}_2F_1\left(\frac{7}{4} + \frac{i}{2}\nu, \frac{7}{4} - \frac{i}{2}\nu; \frac{3}{2}; -x\right) \right]. \end{aligned} \quad (6.18)$$

Using formulae for the integration of a hypergeometric function times a power function, we find that the integral on the right hand side of (6.18) vanishes, and we are left with a simple expression for β_ν . As a result, $b_{TT}(r, y)$ reads

$$b_{TT}(r, y) = -4G_5 M k \ddot{A}(0) \int_0^\infty \frac{d\nu}{\pi} \frac{C_\nu^2}{\nu^2 + \frac{1}{4}} {}_2F_1\left(\frac{7}{4} + \frac{i}{2}\nu, \frac{7}{4} - \frac{i}{2}\nu; \frac{5}{2}; -k^2 r^2\right) \frac{\psi_\nu(y)}{\psi_\nu(0)}. \quad (6.19)$$

The corresponding expressions for a_{TT} and c_{TT} follow from the traceless and transverse conditions. Then, by substituting the expressions back into (6.3), we obtain the explicit solution to the linearized Einstein equations.

To relate the integral representation found above to the summation over Kaluza-Klein modes, we convert (6.19) into a contour integral

$$\begin{aligned} b_{TT}(r, y) = \frac{8G_5 M k}{3\sqrt{\pi}} \ddot{A}(0) \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{dE_0}{2\pi i} \frac{E_0 \Gamma^2(E_0/2)}{(E_0 - 1)(E_0 - 2)\Gamma(E_0 - \frac{3}{2})} (kr)^{-E_0-2} \\ \times {}_2F_1\left(1 + \frac{1}{2}E_0, -\frac{1}{2} + \frac{1}{2}E_0; -\frac{1}{2} + E_0; -(kr)^{-2}\right) \frac{\psi_{i(E_0 - \frac{3}{2})}(y)}{\psi_{i(E_0 - \frac{3}{2})}(0)}, \end{aligned} \quad (6.20)$$

where we have used the expression for C_ν^2 given in Eq. (6.10). To highlight the $r \rightarrow \infty$ behavior of b_{TT} , we have in addition used the relation between hypergeometric functions of arguments ζ and $1/\zeta$. Enclosing the contour from the right half plane, we pick up the

residues of the poles along the positive real axis (see Fig. 1). In addition to the trapped graviton and Kaluza-Klein modes (previously identified in Sections 3 and 4) which are specified by the condition $\dot{\psi}_{i(E_0 - \frac{3}{2})}(0) = 0$ (which is simply the Israel matching condition), we find a pole at $E_0 = 2$ corresponding to the presence of a radion mode; its residue produces exactly the term of (6.6) proportional to $b_{BR}(r)\psi_2(y)$.

Asymptotic expressions on the brane

To get a better understanding of the integral representation for b_{TT} , we turn to several asymptotic limits. We first explore the Randall-Sundrum limit by taking $k \rightarrow 0$ [or equivalently $y_0 \rightarrow \infty$, as seen from (2.9)], while keeping $\kappa y = \mathcal{O}(1)$. Introducing $p = \nu k$ and $\zeta = e^{\kappa y}/\kappa$, we note that

$$\tanh \kappa(y - y_0) \simeq -1 + \frac{k^2 \zeta^2}{2}. \quad (6.21)$$

It follows that the behavior of the hypergeometric representation (6.11) of $\psi_\nu(y)$ singles out $p\zeta = \mathcal{O}(1)$. Highlighting this behavior motivates us to split the range of integration of Eq. (6.19) according to

$$\int_0^\infty \frac{d\nu}{\pi} = \int_0^{\nu_0} \frac{d\nu}{\pi} + \int_{\nu_0}^\infty \frac{d\nu}{\pi} = \int_0^{\nu_0} \frac{d\nu}{\pi} + \frac{1}{k} \int_{k\nu_0}^\infty \frac{dp}{\pi},$$

where ν_0 is fixed, and satisfies $1 \ll \nu_0 \ll \kappa/k$. Correspondingly, we write

$$b_{TT}(r, y) = b_{TT}^{LL}(r, y) + b_{TT}^{RS}(r, y). \quad (6.22)$$

We now treat b_{TT}^{RS} and b_{TT}^{LL} separately, according to the range of integration.

To estimate the contribution of b_{TT}^{LL} , we approximate the hypergeometric function in $kr \ll 1$ of the integrand by unity and

$$\psi_\nu(y) \simeq \frac{32}{\Gamma(\frac{5}{2} + i\nu)\Gamma(\frac{5}{2} - i\nu)} [1 - \frac{1}{4}(\nu^2 + \frac{9}{4})k^2\zeta^2]. \quad (6.23)$$

These Gamma functions near the upper limit may be estimated with the Stirling formula. The result is that

$$b_{TT}^{LL}(r, y) \simeq \frac{16}{3\pi} \ddot{A}(0) \kappa \nu_0. \quad (6.24)$$

The cutoff ν_0 can be sent to zero in the limit $k \rightarrow 0$.

For b_{TT}^{RS} , on the other hand, the hypergeometric functions in the integrand are dominated by their confluent limits. Following the steps outlined in Appendix A, we find that

$${}_2F_1(\frac{7}{4} + \frac{i}{2}\nu, \frac{7}{4} - \frac{i}{2}\nu; \frac{5}{2}; -k^2 r^2) \simeq \frac{3}{pr} j_1(pr), \quad (6.25)$$

while

$$\psi_\nu(y) \simeq \frac{16}{\pi\nu^2} \zeta^2 K_2(p\zeta) \cosh \pi\nu. \quad (6.26)$$

Combining these expressions, we find that in the Randall-Sundrum limit $b_{TT}(r, y)$ becomes

$$b_{TT}^{RS}(r, y) = \frac{8G_5 M}{3\pi} e^{2\kappa y} \int_0^\infty dp p \frac{j_1(pr)}{pr} \frac{K_2(p\zeta)}{K_1(p/\kappa)}, \quad (6.27)$$

which agrees with the expression obtained previously in Ref. [9].

In the region where $1/k \gg r \gg 1/\kappa$ on the brane, the integral (6.27), for b_{TT}^{RS} may be approximated. The result is

$$b_{TT}^{RS}(r, 0) \simeq \frac{4G_4 M}{3r} + \frac{4G_4 M}{3\kappa^2 r^3} (\ln 2\kappa r - 1), \quad (6.28)$$

where we have used the two-sided Randall-Sundrum relation $G_4 = \kappa G_5$. We must take one more step before examining the geometry, and that is to account for the b_{BR} mode, which in the $k \rightarrow 0$ limit reads $b_{BR}(r) \simeq 4G_5 M / 3k^2 r^3$. Substituting b_{TT} and b_{BR} into (6.3), and using $\psi_1(0) \simeq -\kappa$ as well as the transverse-traceless conditions to recover a_{TT} and c_{TT} , we find

$$\begin{aligned} a &\simeq -\frac{4G_4 M}{3r} \left(1 + \frac{1}{\kappa^2 r^2} \right), \\ b &\simeq \frac{4G_4 M}{3r} \left(1 - \frac{1}{k^2 r^2} + \frac{1}{\kappa^2 r^2} (\ln 2\kappa r - 1) \right), \\ c &\simeq \frac{2G_4 M}{3r} \left(\frac{1}{k^2 r^2} - \frac{1}{\kappa^2 r^2} (\ln 2\kappa r - 2) \right). \end{aligned} \quad (6.29)$$

By performing a gauge transformation to set $c = 0$, the metric on the brane is finally transformed into the standard form

$$ds^2 = - \left[1 - \frac{2G_4 M}{r} \left(1 + \frac{2L_5^2}{3r^2} \right) \right] dt^2 + \left[1 + \frac{2G_4 M}{r} \left(1 + \frac{L_5^2}{r^2} \right) \right] dr^2 + r^2 d\Omega^2, \quad (6.30)$$

which is simply the linearized Schwarzschild solution including the first Randall-Sundrum correction [2,5]. This reproduces the Newtonian potential, (1.1).

We now return to the Karch-Randall case, and examine the large distance behavior, $kr \gg 1$. The large r behavior of $b_{TT}(r, y)$ is easily obtained through its contour integral representation, (6.20), which enables us to write

$$b_{TT}(r, y) = \bar{b}_{\text{radion}}(r, y) + b_0(r, y) + \sum b_{KK}(r, y). \quad (6.31)$$

This is the decomposition as a sum over residues of the poles in (6.20). Here, $\bar{b}_{\text{radion}}(r, y) \sim b_{BR}(r)\psi_2(y)$ is the $E_0 = 2$ residue corresponding to the physical radion, $b_0(r, y)$ is the

$E_0 \approx 3$ quasi-zero mode discussed in Section 3, and b_{KK} represents the $E_0 \approx 4, 5, 6, \dots$ Kaluza-Klein modes discussed in Section 4.

For large kr , the quasi-zero mode takes the form

$$b_0(r, 0) \simeq -\frac{4G_5 M \kappa}{3k} \ddot{A}(0) (kr)^{-5-k^2/2\kappa^2}, \quad (6.32)$$

while the sum over the KK modes is dominated by its first term, which corresponds to the pole at $E_0 \simeq 4 + 3k^2/2\kappa^2$. Retaining only the leading term, we have

$$\sum b_{KK}(r, 0) \simeq -\frac{16G_5 M k}{9\pi \kappa} \ddot{A}(0) (kr)^{-6-3k^2/2\kappa^2} + \dots \quad (6.33)$$

In general, it is clear from (6.20) that the large kr behavior of the residue at the pole for E_0 is simply $b_{E_0}(r, 0) \sim (kr)^{-2-E_0}$. So in fact the large kr behavior of $b(r, 0)$ itself is dominated by that of the radion mode

$$b_{\text{radion}}(r, 0) \simeq \frac{4G_5 M k}{3\pi \kappa} \ddot{A}(0) [(kr)^{-4} - \frac{2}{3}(kr)^{-6} + \dots]. \quad (6.34)$$

Combining the above expressions, this gives for $b(r, 0)$

$$b(r, 0) \simeq \frac{4G_4 M k}{3} \left[\frac{1}{\pi} \left(\frac{k}{\kappa} \right)^2 (kr)^{-4} - (kr)^{-5-k^2/2\kappa^2} - \frac{4}{3\pi} \left(\frac{k}{\kappa} \right)^2 (kr)^{-6-3k^2/2\kappa^2} + \dots \right]. \quad (6.35)$$

Finally, we recover $a(r, 0)$ and $c(r, 0)$ from the transverse-traceless conditions and perform a gauge transformation to set $c(r, 0) = 0$. This yields the linearized Karch-Randall solution

$$\begin{aligned} a &\simeq -2G_4 M k \left[(kr)^{-3-k^2/2\kappa^2} + \frac{8}{3\pi} \left(\frac{k}{\kappa} \right)^2 (kr)^{-4-3k^2/2\kappa^2} + \dots \right], \\ b &\simeq 2G_4 M k \left[(kr)^{-3-k^2/2\kappa^2} - \frac{2}{9\pi} \left(\frac{k}{\kappa} \right)^2 (kr)^{-4} + \frac{32}{9\pi} \left(\frac{k}{\kappa} \right)^2 (kr)^{-4-3k^2/2\kappa^2} + \dots \right]. \end{aligned} \quad (6.36)$$

This series representation generalizes the metric obtained from the quasi-zero mode, (3.18), to include the contributions from the radion and KK modes. However, unlike in the Randall-Sundrum case, (6.30), where only every other power of r enters the metric, here all integer powers plus fractional mass shifts contribute.

We note that the radion first enters a at $\mathcal{O}(r^{-6})$, and even in b is no longer dominant over the quasi-zero mode graviton. Nevertheless, the radion does contribute at subleading order, and its omission in b would lead to an incorrect prediction of the Karch-Randall corrections to Schwarzschild-AdS. Of course, this is mainly of academic interest, as the expansion of (6.36) is only valid in the $kr \gg 1$ regime. Furthermore, it is important to realize that the presence of the radion does not lead to a VVZ discontinuity, as the conventional Randall-Sundrum metric, (6.30), is obtained smoothly in the $k \rightarrow 0$ limit.

7. Conclusions.

In this paper, we have investigated the metric fluctuations of the Karch-Randall model in a linearized gravity framework. While our approach has mainly focused on the properties of a static, axially symmetric metric in $D = 4 + 1$ dimensions giving rise to an AdS_4 braneworld, much of the analysis of the Kaluza-Klein modes is independent of the metric on the brane, and is hence completely general (at least within the limitations of linearized gravity). In first examining the source-free (on the brane) Einstein equations, we presented an elementary procedure for obtaining the mass of the (quasi-zero mode) Karch-Randall graviton. Despite its mass, we demonstrate that the presence of this massive graviton continues to reproduce the Schwarzschild- AdS_4 metric at large distances on the brane.

Of course, the notion of mass in anti-de Sitter space is rather different from that in asymptotic Minkowski space. In AdS , the large distance behavior, $r \gg L$ (or, equivalently, $kr \gg 1$) is governed primarily by the AdS curvature. Thus wavefunctions naturally fall off extremely rapidly at large distances, regardless of mass. In the reference AdS_4 background, (2.10), the falloff is essentially given by r^{-E_0} , as may be seen in the power behavior of (6.36). A more directly observable effect of mass (*i.e.* Yukawa behavior) presumably may be seen only within an AdS radius. However, for a Karch-Randall graviton of mass (1.2), with Compton wavelength quadratic in L , this mass is essentially vanishing, and presumably there is no regime where Yukawa behavior may be observed. For this same reason, we believe a massive graviton as given here does not signify a new screened phase of gravity, regardless of how general covariance is broken (or whether it is even broken, as in the dynamical mass generation mechanism of Refs. [23,24]). This protection against screening is safe in the Minkowski limit, as any finite E_0 leads to vanishing mass when the cosmological constant is removed.

A somewhat unexpected result of this linearized gravity analysis is the discovery of a physical radion in the (one brane) Karch-Randall model. The radion is an $E_0 = 4$ scalar mode coupling to the trace of the stress-energy tensor on the brane, and gives rise to a physical correction to the Schwarzschild- AdS metric. In the present case, the radion arises through an incomplete cancellation of the $E_0 = 4$ scalar fluctuation with a similar brane-bending mode (and would be absent in the RSII model, where the cancellation is complete). While the radion is directly sourced by matter on the brane, we have demonstrated that it is always present in the spectrum, even in the absence of sources. Although this analysis was only performed for static configurations (and in addition with axial symmetry), the existence of the radion mode may be traced to solutions of the KK mode equation (2.22), and hence is insensitive to any particular metric ansatz on the brane. This suggests that the radion survives as a completely dynamical model on the brane, and not simply as a constrained trace mode of the graviton.

In order to explain the physical origin of the Karch-Randall radion, we note that unlike in the Randall-Sundrum model, where the position of the brane may be removed by

a conformal rescaling (and hence is unphysical), for the AdS braneworld, the distance y_0 from the brane to the minimum of the double exponential warp factor is not easily removed. Only in the Randall-Sundrum limit, when y_0 is pushed to infinity, does this special point disappear. It is the promotion of this y_0 distance to a dynamical field that gives rise to a radion. In this way, this feature is similar to the RSI model, where the distance between the two branes is governed by a radion (which similarly disappears when the second brane is removed to infinity). Of course, one may argue that there is no preferred point such as y_0 in the bulk AdS₅ space. While this is certainly the case, it is important to keep in mind that the braneworld is not given simply AdS₅, but depends crucially on how it is sliced up. After all, it is this difference that gives rise to distinct classes of dS, flat and AdS braneworlds, which are all embedded in AdS₅. An important aspect of this slicing of AdS would be the identification of the radion in a holographic interpretation of the Karch-Randall model.

In the present analysis, it is unclear whether this radion couples as a ghost or not. However, the large distance expansion of (6.36) is suggestive of a ghost, as the radion enters with the “wrong” sign in the metric correction. A determination of whether the radion is a ghost or not would be of crucial significance to the (classical) stability of the Karch-Randall braneworld. This is currently under investigation in a more general braneworld setting, although still within the linearized gravity framework.

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Appendix A.

In section 4, we made use of several analytic properties of hypergeometric functions. Here we summarize some of the technical issues that were involved. Firstly, for the off-brane wavefunction (4.6), the analytical continuation of ${}_2F_1(\alpha, 5 - \alpha; 3; \zeta)$ to the neighborhood of $\zeta = 1$ is given by [33]

$$\begin{aligned} {}_2F_1(\alpha, 5 - \alpha; 3; \zeta) &= \frac{2}{\Gamma(\alpha)\Gamma(5 - \alpha)} \left[\frac{1}{(1 - \zeta)^2} + \frac{(2 - \alpha)(3 - \alpha)}{(1 - \zeta)} \right] \\ &+ \frac{2}{\Gamma(3 - \alpha)\Gamma(-2 + \alpha)} \sum_{l=0}^{\infty} \frac{(\alpha)_l (5 - \alpha)_l}{l!(l + 2)!} \left[\psi(l + 1) + \psi(l + 3) \right. \\ &\left. - \psi(\alpha + l) - \psi(5 - \alpha + l) - \ln(1 - \zeta) \right] (1 - \zeta)^l. \end{aligned} \quad (\text{A.1})$$

Using the relation

$$\Gamma(\zeta)\Gamma(1-\zeta) = \frac{\pi}{\sin \pi \zeta}, \quad (\text{A.2})$$

and its logarithmic derivative

$$\psi(\zeta) - \psi(1-\zeta) = -\pi \cot \pi \zeta, \quad (\text{A.3})$$

Eq. (1.1) can be written as

$${}_2F_1(-n, 5+n; 3; \zeta) = {}_2F_1(-n, 5+n; 3; 1-\zeta) \cos n\pi + G(-n, 5+n; 3; 1-\zeta) \sin n\pi, \quad (\text{A.4})$$

where

$$\begin{aligned} G(-n, 5+n; 3; 1-\zeta) = & -\frac{2}{\pi} \left[\frac{1}{(n+1)(n+2)(n+3)(n+4)(1-\zeta)^2} + \frac{1}{(n+1)(n+2)(1-\zeta)} \right] \\ & -\frac{2}{\pi} \sum_{l=0}^{\infty} \frac{(-n)_l (5+n)_l}{l!(l+2)!} \left[\psi(l+1) + \psi(l+3) \right. \\ & \left. - \psi(n-l+1) - \psi(5+n+l) - \ln(1-\zeta) \right] (1-\zeta)^l. \end{aligned} \quad (\text{A.5})$$

The order by order approximation for small $1-\zeta$ at fixed n can be obtained readily, thus giving rise to Eq. (4.7).

Secondly, to obtain (4.9), we consider the series representation of ${}_2F_1(-n, 5+n; 3; 1-\zeta)$ with $1-\zeta = \xi^2/4n^2$:

$${}_2F_1(-n, 5+n; 3; \frac{\xi^2}{4n^2}) = \sum_{l=0}^{\infty} \frac{(-n)_l (5+n)_l}{l!(l+2)!} \left(\frac{\xi}{2n} \right)^{2l}. \quad (\text{A.6})$$

As $n \rightarrow \infty$, the contribution of the successive terms drops quickly before l becomes comparable to n . Therefore the leading order approximation amounts to setting $(-n)_l \simeq (-n)^l$ and $(5+n)_l \simeq n^l$. The series (1.6) then becomes proportional to that of the Bessel function J_2 :

$${}_2F_1(-n, 5+n; 3; \frac{\xi^2}{4n^2}) \simeq 2 \left(\frac{2}{\xi} \right)^2 J_2(\xi). \quad (\text{A.7})$$

Using the same approximation, we may set in addition $\psi(n-l+1) \simeq \psi(5+n+l) \simeq \ln n$. Hence the series representation of $G(-n, 5+n; 3; \frac{\xi^2}{4n^2})$ becomes

$$G(-n, 5+n; 3; \frac{\xi^2}{4n^2}) \simeq 2 \left(\frac{2}{\xi} \right)^2 N_2(\xi). \quad (\text{A.8})$$

The wavefunction for the higher modes, (4.9), is easily obtained by substituting (1.7) and (1.8) into (1.4). Finally, working out the confluent limit of the function $b(r|\mu)$ to arrive at (6.25) follows in a similar manner.

Appendix B.

In this appendix, we derive the generalized Fourier transformation utilized in section 6. For proper normalization, we begin by introducing a large box with

$$0 < r < \frac{1}{k}e^L. \quad (\text{B.1})$$

Furthermore, we impose Dirichlet boundary conditions on function $u(r|\mu)$, *i.e.*

$$u\left(\frac{1}{k}e^L|\mu\right) = 0. \quad (\text{B.2})$$

The spectrum ν^2 becomes discrete and is determined by the condition

$$\nu L + \delta_\nu = n\pi, \quad n = 0, 1, 2, \dots, \quad (\text{B.3})$$

for $\nu L \gg 1$. Using the asymptotic formula, we find the normalization integral

$$\int_0^{\frac{1}{k}e^L} dr r^4 (1 + k^2 r^2) u^2(r|\mu) = \frac{L}{C_\nu^2 k^5}. \quad (\text{B.4})$$

The normalized eigenfunction now reads

$$\hat{u}(r|\mu) = \frac{C_\nu k^{\frac{5}{2}}}{\sqrt{L}} {}_2F_1\left(\frac{7}{4} + \frac{i}{2}\nu, \frac{7}{4} - \frac{i}{2}\nu; \frac{5}{2}; -k^2 r^2\right). \quad (\text{B.5})$$

It follows from Sturm-Liouville theory that $\hat{u}(r|\mu)$ forms a complete basis set according to which any function can be expanded:

$$f(r) = \frac{1}{\sqrt{L}} \sum_\nu B_\nu \hat{u}(r|\mu), \quad (\text{B.6})$$

and

$$B_\nu = \sqrt{L} \int_0^\infty dr r^4 (1 + k^2 r^2) \hat{u}(r|\mu) f(r). \quad (\text{B.7})$$

In the limit $L \rightarrow \infty$, $\sum_\nu \rightarrow \frac{L}{\pi} \int_0^\infty d\nu$, and Eqs. (2.6) and (2.7) become the generalized Fourier transformation, (6.14) and (6.15), of section 5.

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Figures.

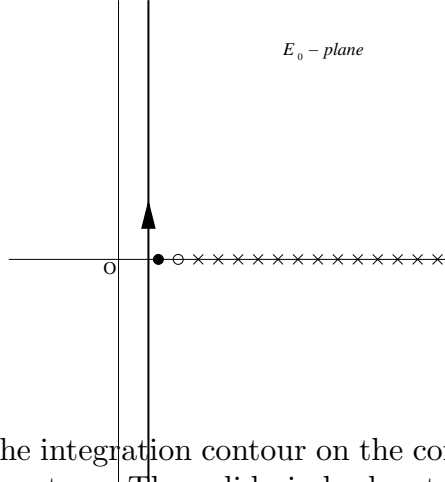


Figure 1: The integration contour on the complex E_0 -plane and the poles to the right of the contour. The solid circle denotes the radion mode, the open circle represents the quasi-zero mode and the crosses the KK modes.